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# The black hole interior and a curious sum rule

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**ABSTRACT:** We analyze the Euclidean geometry near non-extremal NS5-branes in string theory, including regions beyond the horizon and beyond the singularity of the black brane. The various regions have an exact description in string theory, in terms of cigar, trumpet and negative level minimal model conformal field theories. We study the worldsheet elliptic genera of these three superconformal theories, and show that their sum vanishes. We speculate on the significance of this curious sum rule for black hole physics.

**KEYWORDS:** Black Holes in String Theory, Conformal Field Models in String Theory

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## 1 Introduction

The Wick rotation of the Lorentzian black hole geometry to a Euclidean solution of gravity gives valuable information about quantum black holes. For the Schwarzschild solution, the Euclidean geometry reads

$$ds^2 = (1 - 2M/r)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega_2^2. \quad (1.1)$$

The region that corresponds to the exterior of the black hole  $r > 2M$  is known as the cigar geometry. From the cigar geometry, one reads off the relationship between the black hole temperature and mass and the value of the Euclidean action that corresponds to the entropy of the black hole [1], and one can fix the Hartle-Hawking wave function [2] associated with the Lorentzian extended geometry [3].

The region that corresponds to the interior of the black hole,  $r < 2M$ , does not appear to be physical. In the region between the horizon and the singularity,  $0 < r < 2M$ , the space-time signature is  $(-, -, +, +)$ . It includes two time directions ( $r$  and  $t$ ). The big advantage of Wick rotation is thus lost; instead of getting a Euclidean space in which the path integral is well defined, we get a divergent path integral. The region beyond the singularity  $r < 0$  is again more standard in that its signature is Euclidean.

In this short note, we study an analogue of the Schwarzschild geometry which is the geometry near non-extremal NS5-branes. We consider the three regions identified above, after analytic continuation, and study those regions from the perspective of a string worldsheet theory. We establish a curious sum rule for the worldsheet elliptic genera associated to the three Euclidean regions. This could be viewed as an indication that in string theory the interior of the black hole plays an important role even after Wick rotation.

## 2 The NS5-brane geometry

The near horizon Euclidean geometry associated with  $k$  near extremal NS5-branes in the type II superstring is described (at  $\alpha' = 2$ ) by the metric and dilaton [4]:

$$ds^2 = (1 - 2M/r)dt^2 + \frac{k}{2r^2} \frac{dr^2}{1 - 2M/r} + 2kd\Omega_3^2 + ds_{T^5}^2, \\ e^{-2\Phi} = \frac{r}{2k}, \quad (2.1)$$

where  $M$  is the energy density above extremality. There is an  $H$ -field flux on the three-sphere corresponding to  $k$  NS5-branes. The geometry has a similar causal structure to the Schwarzschild geometry (1.1).

In a first region I, outside the horizon  $r > 2M$ , we have a two-dimensional cigar geometry (times an  $SU(2)_k$  Wess-Zumino-Witten model, a five-dimensional flat space, as well as a non-trivial dilaton profile). It has an exact conformal field theory description in terms of an  $SL(2, \mathbb{R})_k/U(1)$  coset conformal field theory [5–7].

In a second region II, between the horizon and singularity,  $0 < r < 2M$ , we obtain a bell geometry with all negative signature, corresponding to an  $SU(2)_{-k}/U(1)$  coset conformal field theory at a negative level  $-k$ . The fact that the level is negative, and that we therefore have two time directions is also dictated by the vanishing of the total central charge of the string theory. The seeming singularity in the string coupling and metric are absent in the exact conformal field theory description (see e.g. [8, 9] for detailed discussions).

Finally, region III, the region beyond the singularity, is obtained by setting  $r < 0$ . The resulting geometry is that of the trumpet, the vectorially gauged  $SL(2, \mathbb{R})_k/U(1)$  coset conformal field theory (see e.g. [10] for a review). See figure 1 for a map of the regions in the Lorentzian black hole in Kruskal coordinates to the conformal field theory target spaces.

For the regions outside the horizon and beyond the singularity, there are exact conformal field theory descriptions. The elliptic genera of these theories are known [11–13]. It should be noted that the trumpet theory is T-dual to the  $\mathbb{Z}_k$  orbifold of the cigar (see e.g. [10]). The region between the horizon and the singularity corresponds to an  $N = 2$  minimal model at a negative level. The elliptic genus of the conformal field theory with positive level was calculated longer ago [14]. We analytically continue the result into negative level  $-k$  and we demonstrate below the curious fact that the sum of the three elliptic genera vanishes.

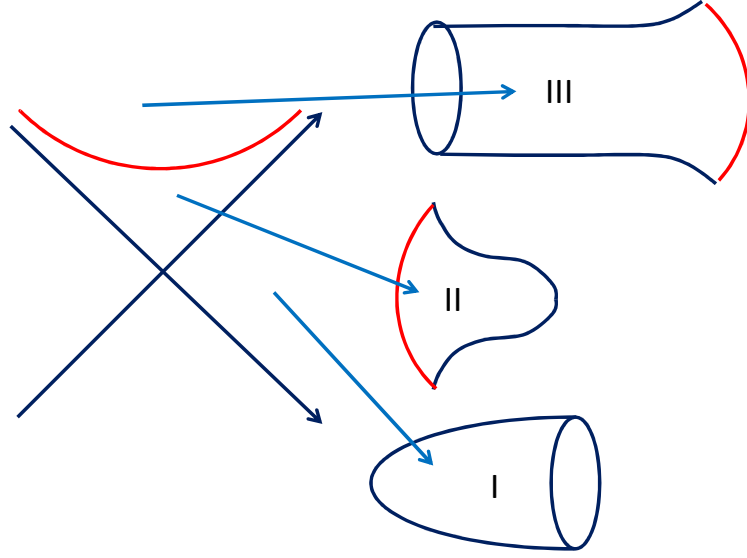
## 3 The sum rule

The identity we wish to prove reads:

$$\chi_I + \chi_{II} + \chi_{III} \equiv \chi_{\text{cos}}(k) + \chi_{\text{MM}}(-k) + \chi_{\text{orb}}(k) = 0, \quad (3.1)$$

where the minimal model elliptic genus  $\chi_{\text{MM}}$  at level  $k$  is given by:

$$\chi_{\text{MM}}(k) = \frac{\theta_{11}(q, z^{1-\frac{1}{k}})}{\theta_{11}(q, z^{\frac{1}{k}})}, \quad (3.2)$$



**Figure 1.** The relationship between the Lorentzian black hole in Kruskal coordinates and the different regions obtained after Wick rotation. Regions *I* and *III* are Euclidean. Region *II* includes two time directions.

and the cigar coset and  $\mathbb{Z}_k$  orbifold elliptic genera  $\chi_{\text{cos}}$  and  $\chi_{\text{orb}}$  are given by:

$$\begin{aligned}
 \chi_{\text{cos}} &= \chi_{\text{cos}}^{\text{hol}} + \chi_{\text{cos}}^{\text{rem}} \\
 \chi_{\text{cos}}^{\text{hol}} &= \frac{i\theta_{11}(q, z)}{\eta^3} \sum_{\gamma=0}^{k-1} \sum_{m \in \mathbb{Z}} \frac{q^{km^2 - m\gamma} z^{2m - \frac{\gamma}{k}}}{1 - zq^{km - \gamma}} \\
 \chi_{\text{cos}}^{\text{rem}} &= -\frac{i\theta_{11}(q, z)}{\pi\eta^3} \sum_{n, w} \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} \frac{ds}{2is + n + kw} q^{\frac{s^2}{k} + \frac{(n - kw)^2}{4k}} z^{\frac{kw - n}{k}} \bar{q}^{\frac{s^2}{k} + \frac{(n + kw)^2}{4k}} \\
 \chi_{\text{orb}} &= \chi_{\text{orb}}^{\text{hol}} + \chi_{\text{orb}}^{\text{rem}} \\
 \chi_{\text{orb}}^{\text{hol}} &= \frac{i\theta_{11}(q, z)}{\eta^3} \sum_{m \in \mathbb{Z}} \frac{q^{km^2} z^{2m}}{1 - z^{\frac{1}{k}} q^m} \\
 \chi_{\text{orb}}^{\text{rem}} &= -\frac{i\theta_{11}(q, z)}{\pi\eta^3} \sum_{n, w} \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} \frac{ds}{2is + n + kw} q^{\frac{s^2}{k} + \frac{(n - kw)^2}{4k}} z^{\frac{n - kw}{k}} \bar{q}^{\frac{s^2}{k} + \frac{(n + kw)^2}{4k}}. \quad (3.3)
 \end{aligned}$$

The proof of the identity proceeds as follows. In a first step, we notice that the first and third term in equation (3.1) combine into a holomorphic expression. This is because of the identity (see the appendix of [15]):

$$\chi_{\text{cos}}^{\text{rem}} = -\frac{i\theta_{11}}{\eta^3} \sum_{m \in \mathbb{Z}} q^{km^2} z^{-2m} - \chi_{\text{orb}, \text{rem}}. \quad (3.4)$$

The second step in the proof uses the theory of elliptic functions, or the theory of Jacobi forms. In particular, we follow a proof of the fact that the sum of  $N = 2$  minimal model

characters and a ratio of theta-functions are equal [16]. Thus, we attempt to prove that the ratio of the two elliptic functions is one:

$$\frac{\chi_{\text{cos}} + \chi_{\text{orb}}}{-\chi_{\text{MM}}(-k)} = 1. \quad (3.5)$$

We observe that both numerator and denominator have identical modular and elliptic transformations properties. The ratio is therefore an elliptic function of  $z^k$ , and a modular form of weight zero. Moreover both numerator and denominator are holomorphic, due to the reasoning in step one of the proof. By inspection, it is clear that there are a finite number of poles and zeroes in the fundamental domain. It is therefore a ratio of a finite number of theta functions, which equals to one on the condition that the expression equals one in the  $q \rightarrow 0$  limit [16]. This limit is easily taken, and the condition is verified. Therefore, we have proven the identity.

**Side remark.** Note that in the special case of level  $k = 1$  our identity encompasses a relation between the conifold and an analytically continued minimal model elliptic genus [17, 18]. It explains the relative factor of  $\frac{1}{2}$  between the two, which arises from the fact that at level  $k = 1$ , the cigar and its  $\mathbb{Z}_k$  orbifold are identical. Our identity also correctly captures the overall sign. The latter can be fixed from the Witten indices of the individual models.

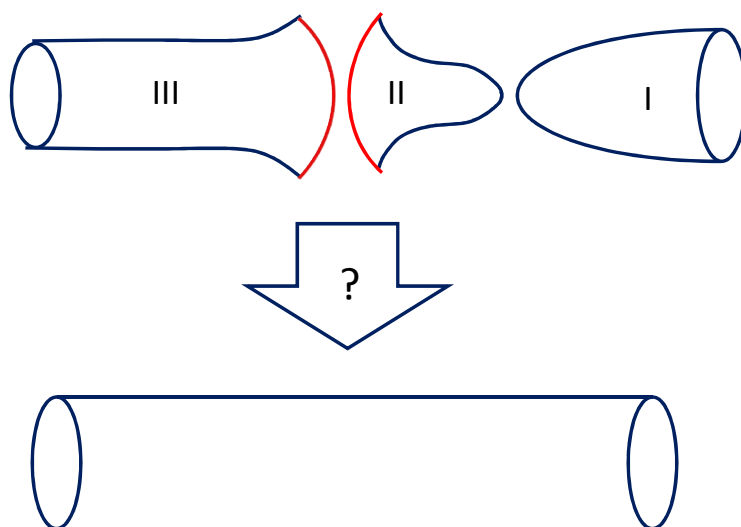
## 4 Interpretations

A prosaic interpretation of the sum rule is that it is merely a curious mathematical fact. Indeed, the elliptic genus is not an observable in string theory. A second possibility is that this identity is a consequence of the local physics near the cigar tip, the tip of the orbifolded cigar, and the T-fold interpretation of the minimal model geometry [8, 9]. An exciting possibility is that the identity provides us with an important hint about black hole physics. The elliptic genus does provide us with information concerning the states of the conformal field theories underlying the various regimes, and consequently the content of states in the full fledged string theory on the black-hole background.

Let us entertain the possibility then that we obtained a hint on black hole physics. We have a sum of three different conformal field theory elliptic genera that vanishes. One can view the zero on the right hand side of the sum rule (3.1) as the elliptic genus of a conformal field theory with a vanishing elliptic genus. A natural guess for such a conformal field theory is the conformal field theory on the cylinder. Namely, the sum rule should be read (see figure 2):

$$\chi_{\text{I}} + \chi_{\text{II}} + \chi_{\text{III}} = \chi(\text{cylinder}). \quad (4.1)$$

If so, one can view the identity as an indication that there is a space associated to the direct sum of the three conformal field theories associated to the three Euclidean regions of the black brane geometry, that is topologically a cylinder. While the two time directions in the region between the horizon and the singularity render an intuitive interpretation hard, we believe that the point of view in which one takes the region beyond the horizon seriously even in the Euclidean setting needs to be further explored.



**Figure 2.** Equation (4.1) suggests that the topology of the combined regions is that of a cylinder.

**Note added.** The identity (3.1) was independently found and very recently published in [19], where an interpretation in terms of a compactification of the cigar conformal field theory was advanced.

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## References

- [1] G. Gibbons and S. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev. D* **15** (1977) 2752 [[INSPIRE](#)].
- [2] J.B. Hartle and S.W. Hawking, *Path integral derivation of black hole radiance*, *Phys. Rev. D* **13** (1976) 2188 [[INSPIRE](#)].
- [3] W. Israel, *Thermo field dynamics of black holes*, *Phys. Lett. A* **57** (1976) 107 [[INSPIRE](#)].
- [4] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys. B* **360** (1991) 197 [[INSPIRE](#)].

- [5] S. Elitzur, A. Forge and E. Rabinovici, *Some global aspects of string compactifications*, *Nucl. Phys. B* **359** (1991) 581 [INSPIRE].
- [6] G. Mandal, A.M. Sengupta and S.R. Wadia, *Classical solutions of two-dimensional string theory*, *Mod. Phys. Lett. A* **6** (1991) 1685 [INSPIRE].
- [7] E. Witten, *On string theory and black holes*, *Phys. Rev. D* **44** (1991) 314 [INSPIRE].
- [8] J.M. Maldacena, G.W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *JHEP* **07** (2001) 046 [hep-th/0105038] [INSPIRE].
- [9] C. Kounnas, N. Toumbas and J. Troost, *A wave-function for stringy universes*, *JHEP* **08** (2007) 018 [arXiv:0704.1996] [INSPIRE].
- [10] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [hep-th/9401139] [INSPIRE].
- [11] J. Troost, *The non-compact elliptic genus: mock or modular*, *JHEP* **06** (2010) 104 [arXiv:1004.3649] [INSPIRE].
- [12] T. Eguchi and Y. Sugawara, *Non-holomorphic modular forms and  $SL(2, \mathbb{R})/U(1)$  superconformal field theory*, *JHEP* **03** (2011) 107 [arXiv:1012.5721] [INSPIRE].
- [13] S.K. Ashok and J. Troost, *A twisted non-compact elliptic genus*, *JHEP* **03** (2011) 067 [arXiv:1101.1059] [INSPIRE].
- [14] E. Witten, *On the Landau-Ginzburg description of  $N = 2$  minimal models*, *Int. J. Mod. Phys. A* **9** (1994) 4783 [hep-th/9304026] [INSPIRE].
- [15] S.K. Ashok and J. Troost, *Elliptic genera of non-compact Gepner models and mirror symmetry*, *JHEP* **07** (2012) 005 [arXiv:1204.3802] [INSPIRE].
- [16] P. Di Francesco and S. Yankielowicz, *Ramond sector characters and  $N = 2$  Landau-Ginzburg models*, *Nucl. Phys. B* **409** (1993) 186 [hep-th/9305037] [INSPIRE].
- [17] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys. B* **463** (1996) 55 [hep-th/9511164] [INSPIRE].
- [18] T. Eguchi and Y. Sugawara,  *$SL(2, \mathbb{R})/U(1)$  supercoset and elliptic genera of noncompact Calabi-Yau manifolds*, *JHEP* **05** (2004) 014 [hep-th/0403193] [INSPIRE].
- [19] Y. Sugawara, *‘Analytic continuation’ of  $N = 2$  minimal model*, arXiv:1311.4708 [INSPIRE].